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Quantum Optics
Winter semester 2018/2019 - Exercise sheet 4
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Problem 1: Two-mode squeezed states of light.
a) Given the two-mode squeezing operator $\hat{S}(\xi_{12}) = \exp(\xi_{12}^* \hat{a}_1 \hat{a}_2 - \xi_{12} \hat{a}_1^\dagger \hat{a}_2^\dagger)$, show that

$$\hat{S}^\dagger(\xi_{12}) \hat{a}_1 \hat{S}(\xi_{12}) = \mu \hat{a}_1 - \nu \hat{a}_2^\dagger \quad \text{and} \quad \hat{S}^\dagger(\xi_{12}) \hat{a}_2 \hat{S}(\xi_{12}) = \mu \hat{a}_2 - \nu \hat{a}_1^\dagger,$$

where $\mu = \cosh(|\xi_{12}|)$, $\nu = e^{i\phi_\nu} \sinh(|\xi_{12}|)$ and $\xi_{12} = e^{i\phi_\nu}|\xi_{12}|$.

b) For a two-mode radiation field strength $\hat{F} = F_1 \hat{a}_1 + \hat{F}_1^* \hat{a}_1^\dagger + F_2 \hat{a}_2 + \hat{F}_2^* \hat{a}_2^\dagger$, where $F_\lambda = |F_\lambda| \exp(i\phi_\lambda)$, show that the (normal ordered) variance of $\hat{F}$ for the state $\hat{S}(\xi_{12})|0\rangle = |\xi_{12}\rangle$ is given by:

$$\langle \xi_{12}| : (\Delta \hat{F})^2 : |\xi_{12}\rangle = 2(|F_1|^2 + |F_2|^2)|\nu|^2 \left[1 - \frac{2|F_1 F_2|}{|F_1|^2 + |F_2|^2} \sqrt{\frac{1+|\nu|^2}{|\nu|^2}} \cos(\phi_1 + \phi_2 + \phi_\nu) \right].$$

c) Considering $\theta = \phi_1 + \phi_2 + \phi_\nu$ and $|F_1| = |F_2| = 1$, plot this variance as a function of θ for $|\nu| = 1/\sqrt{3}$ and $|\nu| = 1/\sqrt{15}$. How does the squeezing change?

Problem 2: Decomposition of the squeezing operator (alternative derivation).
Assuming that the squeezing operator can be parametrized in two ways as

$$\hat{S}(\lambda) = \exp\left[\frac{1}{2}\lambda\left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right)\right] = \exp\left[\frac{1}{2}\tau_1(\lambda)\hat{a}^{\dagger 2}\right] \exp\left[\tau_2(\lambda)\Big(\hat{n} + \frac{1}{2}\Big)\right] \exp\left[\frac{1}{2}\tau_3(\lambda)\hat{a}^2\right],$$

where $\hat{n} = \hat{a}^\dagger \hat{a}$ and $\lambda \in [0, 1]$, show that

$$\hat{S} \equiv \exp\left[\frac{1}{2}\left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right)\right] = \exp\left[-\frac{\nu}{2\mu}\hat{a}^{\dagger 2}\right] \exp\left[-\ln\mu\,\Big(\hat{n} + \frac{1}{2}\Big)\right] \exp\left[\frac{\nu^*}{2\mu}\hat{a}^2\right],$$

with $\mu = \cosh(|\xi|)$, $\nu = e^{i\phi}\sinh(|\xi|)$ and $\xi = e^{i\phi}|\xi|$.
HINT: use the derivatives of $\hat{S}(\lambda)$ in both forms with respect to λ and compare them. The disentangled form of the squeezing operator allows one to easily see that photons are generated in pairs during the process of parametric downconversion. The operators $\frac{1}{2}\hat{a}^2$ and $\frac{1}{2}\hat{a}^{\dagger 2}$, which appear in the standard form of the single-mode squeezing operator $\hat{S} = \exp\left[\frac{1}{2}\left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right)\right]$, and their commutator $\hat{n} + \frac{1}{2}$ form a basis for the Lie algebra of the $SU(1,1)$ group, to which the squeezing operator belongs. Since the elements of a Lie group can be obtained through exponentiation of the elements of its respective algebra, there are two ways of expressing the elements U of a group: $U(\{\alpha\}) = \exp(\sum_i \alpha_i \Gamma_i)$ or $U(\{\beta\}) = \prod_i \exp(\beta_i \Gamma_i)$, where $\alpha_i \neq \beta_i$, $\{\beta\} = \{\beta_1, \ldots, \beta_N\}$ (and equally for $\{\alpha\}$) and Γ_i are the generators (elements of the basis of the algebra). The transformation of the generators $U(\tau_i)\Gamma_j U^{-1}(\tau_i) = \Gamma_j'$ can be found through differentiation of Γ_j' relative to τ_i.